fore, the noise figure F is given by

$$F = \tfrac{1}{2}(1 + L). \quad (3)$$

This is the same formula as in [5]. Also, the equation relating t, L, and F is

$$t = F/L. \quad (4)$$

Substituting formula (2) into (4) yields

$$t = \frac{1}{L}(1 - n) + n. \quad (5)$$

Measured values for the mixer were

$$L_t = L_r L = 2.09 \qquad (3.2\ \text{dB})$$

$$F_{if} = 1.585 \qquad (2.0\ \text{dB})$$

$$\alpha = 33.0$$

$$F_t = 2.82 \qquad (4.5\ \text{dB}).$$

Calculated values were

$$n = 0.591$$

$$t \cong \frac{1}{L_t}(1 - n) + n = 0.7855$$

$$F_t = 2.86 \quad (4.57\ \text{dB}).$$

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Efficient Numerical Computation of the Frequency Response of Cables Illuminated by an Electromagnetic Field

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Abstract—**Computationally efficient numerical methods for determining the frequency response of uniform transmission lines consisting of a large number of mutually coupled conductors in homogeneous and inhomogeneous media, and illuminated by an electromagnetic (EM) field are presented.**

I. INTRODUCTION

Conductors connecting electronic subsystems on aircraft, missiles, and ground electronic systems are generally grouped into large, closely coupled cable bundles and it is not uncommon to find bundles of over 100 conductors on modern avionics systems (for example, F-4, F-111, and F-15 aircraft) [3]. Determination of the frequency response of these large cable bundles illuminated by high-power radars as well as an electromagnetic pulse (EMP) from nuclear detonations is becoming of increasing importance [1]. Computation of the frequency response of these large bundles illuminated through an aperture such as a landing gear door can be quite costly for only one frequency. However, it is generally necessary to determine the response for many frequencies so the concern here is to minimize the per-frequency computation time for bundles consisting of a large number of conductors. Flat pack and woven flat cables are being used more frequently to connect electronic subsystems and it is not uncommon to find over 35 mutually coupled conductors in these types of cables [7].

Taylor *et al.* [2] considered the problem of two conductors illuminated by a nonuniform electromagnetic (EM) field. For two conductors, the per-frequency computation times are practically minimal. For larger numbers of conductors, we encounter per-frequency computation times which are functions of n^3 for an $(n + 1)$ conductor line so that reduction of the per-frequency computation times becomes an important concern for large numbers of coupled conductors and many computed frequencies.

We will cast the equations to be solved at each frequency into particularly efficient forms as well as introduce computational procedures peculiar to these forms which allow an efficient solution. Perhaps many of the results here will be considered fairly straightforward to obtain but our purposes will be to unify the particular formulations and also point out some perhaps not so obvious techniques for reducing computation times.

Consider an $(n + 1)$ conductor uniform transmission line consisting of $(n + 1)$ parallel lossless conductors of length $\mathcal{L}$ imbedded in a lossless nondispersive medium with the $(n + 1)$st conductor designated as the reference conductor (usually a ground plane or overall shield). The transmission line is described for the TEM mode by the following $2n$ strongly coupled complex differential equations [3], [5]:

$$\begin{bmatrix} \dot{\mathbf{V}}(x) \\ \dot{\mathbf{I}}(x) \end{bmatrix} = -j\omega \begin{bmatrix} {}_n\mathbf{0}_n & \mathbf{L} \\ \mathbf{C} & {}_n\mathbf{0}_n \end{bmatrix} \begin{bmatrix} \mathbf{V}(x) \\ \mathbf{I}(x) \end{bmatrix} \quad (1)$$

where $\dot{\mathbf{V}}(x) = (d/dx)\mathbf{V}(x)$ and ${}_i\mathbf{0}_j$ is the $i \times j$ zero matrix. The distance along the conductor structure and parallel to it is denoted by x; the complex currents $\mathbf{I}(x)$ are directed in the direction of increasing x and the ith elements of the $n \times 1$ vectors $\mathbf{V}(x), V_i(x)$, and $\mathbf{I}(x), I_i(x)$ are the complex potentials (with respect to the reference conductor) and currents, respectively, associated with the ith conductor, $i = 1,\cdots,n$. The parameter ω is the radian frequency of excitation under consideration and the $n \times n$ real symmetric constant matrices $\mathbf{L}$ and $\mathbf{C}$ are the per unit length inductance and capacitance matrices, respectively [3], [5].

The boundary conditions at the ends of the transmission line are in the form of n ports and are characterizable by "generalized Thevenin equivalents" as

$$\mathbf{V}(0) = \mathbf{E}_0 - \mathbf{R}_0\mathbf{I}(0) \quad (2a)$$

$$\mathbf{V}(\mathcal{L}) = \mathbf{E}_{\mathcal{L}} + \mathbf{R}_{\mathcal{L}}\mathbf{I}(\mathcal{L}) \quad (2b)$$

where $\mathbf{E}_0$ and $\mathbf{E}_{\mathcal{L}}$ are $n \times 1$ complex vectors of the equivalent open circuit port excitations and $\mathbf{R}_0$ and $\mathbf{R}_{\mathcal{L}}$ are $n \times n$ real symmetric hyperdominant (and therefore positive definite) matrices representing passive termination networks.

Initially, we must solve (1) and then we must incorporate the boundary conditions of (2). Differentiating the second equation in (1) with respect to x and substituting the first we obtain

$$\ddot{\mathbf{I}}(x) = -\omega^2\mathbf{C}\mathbf{L}\mathbf{I}(x). \quad (3)$$

If we define a change of variables $\mathbf{I}(x) = \mathbf{T}\mathbf{I}_m(x)$ where $\mathbf{T}$ is an $n \times n$ nonsingular matrix and $\mathbf{I}_m(x)$ represent "modal currents" then we obtain

$$\ddot{\mathbf{I}}_m(x) = -\omega^2\mathbf{T}^{-1}\mathbf{C}\mathbf{L}\mathbf{T}\mathbf{I}_m(x). \quad (4)$$

We will show that it is always possible to diagonalize $\mathbf{CL}$ for lines immersed in linear isotropic media and thus uncouple the mode currents $\ddot{\mathbf{I}}_m(x)$ by the similarity transformation $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{C}\mathbf{L}\mathbf{T} = \boldsymbol{\gamma}^2 \quad (5)$$

where $\boldsymbol{\gamma}^2$ is an $n \times n$ diagonal matrix with real positive and nonzero scalars γ_i^2 on the diagonal, i.e., $[\boldsymbol{\gamma}^2]_{ii} = \gamma_i^2$ and $[\boldsymbol{\gamma}^2]_{ij} = 0$ for

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$i \neq j$, $i,j = 1,\cdots,n$ and we denote the element of a matrix $\mathbf{M}$ in the ith row and the jth column by $[\mathbf{M}]_{ij}$. Thus the solution to (3) can be written in terms of $2n$ undetermined constants as [3], [5]

$$\mathbf{I}(x) = \mathbf{T}[\mathbf{e}^{j\omega\gamma x}\boldsymbol{\alpha}^+ + \mathbf{e}^{-j\omega\gamma x}\boldsymbol{\alpha}^-] \tag{6a}$$

and since $\dot{\mathbf{I}}(x) = -j\omega\mathbf{C}\mathbf{V}(x)$ we obtain

$$\mathbf{V}(x) = -\mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma}[\mathbf{e}^{j\omega\gamma x}\boldsymbol{\alpha}^+ - \mathbf{e}^{-j\omega\gamma x}\boldsymbol{\alpha}^-]$$
$$= \mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma}\mathbf{T}^{-1}\{\mathbf{T}[-\mathbf{e}^{j\omega\gamma x}\boldsymbol{\alpha}^+ + \mathbf{e}^{-j\omega\gamma x}\boldsymbol{\alpha}^-]\} \tag{6b}$$

where $\boldsymbol{\alpha}^+$ and $\boldsymbol{\alpha}^-$ are $n \times 1$ vectors of $2n$ undetermined constants and $\mathbf{e}^{j\omega\gamma x}$ is an $n \times n$ diagonal matrix whose entries are $[\mathbf{e}^{j\omega\gamma x}]_{ii} = e^{j\omega\gamma_i x}$ and $[\mathbf{e}^{j\omega\gamma x}]_{ij} = 0$ for $i \neq j$ and $i,j = 1,\cdots,n$.

It is quite natural to define a "characteristic impedance matrix" $\mathbf{Z}_C$ relating the forward and backward waves. From (6) it should be clear that $\mathbf{Z}_C = \mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma}\mathbf{T}^{-1}$. If we write $\mathbf{Y} = j\omega\mathbf{C}$ and $\mathbf{Z} = j\omega\mathbf{L}$ then $\mathbf{Z}_C = \mathbf{Y}^{-1}(\mathbf{YZ})^{1/2}$ which conforms, symbolically, to the scalar characteristic impedance for the two-conductor line and if $\mathbf{CL}$ is diagonalizable by the similarity transformation, $\mathbf{T}$, as in (5), then $(\mathbf{YZ})^{1/2} = j\omega\mathbf{T}\boldsymbol{\gamma}\mathbf{T}^{-1}$.

Incorporating the boundary conditions (2) into (6) one can derive the following matrix equation:

$$\begin{bmatrix} -[\mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma} - \mathbf{R}_0\mathbf{T}] & [\mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma} + \mathbf{R}_0\mathbf{T}] \\ -[\mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma} + \mathbf{R}_\mathcal{L}\mathbf{T}]\mathbf{e}^{j\omega\gamma\mathcal{L}} & [\mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma} - \mathbf{R}_\mathcal{L}\mathbf{T}]\mathbf{e}^{-j\omega\gamma\mathcal{L}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}^+ \\ \boldsymbol{\alpha}^- \end{bmatrix} = \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_\mathcal{L} \end{bmatrix}. \tag{7}$$

The most efficient method of solving m equations in m unknowns is by Gaussian elimination and back substitution (LU decomposition) which requires $(m^3/3 + m^2 - m/3)$ operations (multiplications and divisions) or on the order of $m^3/3$ for large m [6]. Thus the solution of (7) for $\boldsymbol{\alpha}^+$ and $\boldsymbol{\alpha}^-$ at each frequency requires a minimum of $(2n)^3/3$ or $8n^3/3$ operations. Once $\boldsymbol{\alpha}^+$ and $\boldsymbol{\alpha}^-$ are determined from (7), then the potentials and currents at any point on the line can be obtained from (6).

As an alternate approach, consider the use of the matrix chain parameters of the line. The matrix chain parameters can be obtained straightforwardly from (6) by eliminating $\boldsymbol{\alpha}^+$ and $\boldsymbol{\alpha}^-$ to yield

$$\begin{bmatrix} \mathbf{V}(\mathcal{L}) \\ \mathbf{I}(\mathcal{L}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}_{11}(\mathcal{L}) & \boldsymbol{\phi}_{12}(\mathcal{L}) \\ \boldsymbol{\phi}_{21}(\mathcal{L}) & \boldsymbol{\phi}_{22}(\mathcal{L}) \end{bmatrix} \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} = \boldsymbol{\phi}(\mathcal{L}) \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} \tag{8}$$

where the $n \times n$ matrices $\boldsymbol{\phi}_{ij}(\mathcal{L})$ are given by [5]

$$\boldsymbol{\phi}_{11}(\mathcal{L}) = \mathbf{C}^{-1}\mathbf{T}\mathbf{E}^+(\mathcal{L})\mathbf{T}^{-1}\mathbf{C} \qquad \boldsymbol{\phi}_{12}(\mathcal{L}) = -\mathbf{C}^{-1}\mathbf{T}\boldsymbol{\gamma}\mathbf{E}^-(\mathcal{L})\mathbf{T}^{-1}$$
$$\boldsymbol{\phi}_{21}(\mathcal{L}) = -\mathbf{T}\mathbf{E}^-(\mathcal{L})\boldsymbol{\gamma}^{-1}\mathbf{T}^{-1}\mathbf{C} \qquad \boldsymbol{\phi}_{22}(\mathcal{L}) = \mathbf{T}\mathbf{E}^+(\mathcal{L})\mathbf{T}^{-1} \tag{9}$$

and the $n \times n$ diagonal matrices $\mathbf{E}^+(\mathcal{L})$ and $\mathbf{E}^-(\mathcal{L})$ are defined by

$$\mathbf{E}^+(\mathcal{L}) = \tfrac{1}{2}(\mathbf{e}^{j\omega\gamma\mathcal{L}} + \mathbf{e}^{-j\omega\gamma\mathcal{L}}) = \cosh(j\omega\boldsymbol{\gamma}\mathcal{L}) \tag{10a}$$

$$\mathbf{E}^-(\mathcal{L}) = \tfrac{1}{2}(\mathbf{e}^{j\omega\gamma\mathcal{L}} - \mathbf{e}^{-j\omega\gamma\mathcal{L}}) = \sinh(j\omega\boldsymbol{\gamma}\mathcal{L}). \tag{10b}$$

From (8) and the boundary conditions of (2), one can derive in a straightforward manner

$$[\boldsymbol{\phi}_{12}(\mathcal{L}) - \boldsymbol{\phi}_{11}(\mathcal{L})\mathbf{R}_0 - \mathbf{R}_\mathcal{L}\boldsymbol{\phi}_{22}(\mathcal{L}) + \mathbf{R}_\mathcal{L}\boldsymbol{\phi}_{21}(\mathcal{L})\mathbf{R}_0]\mathbf{I}(0) = \mathbf{E}_\mathcal{L} + [\mathbf{R}_\mathcal{L}\boldsymbol{\phi}_{21}(\mathcal{L}) - \boldsymbol{\phi}_{11}(\mathcal{L})]\mathbf{E}_0 \tag{11a}$$

$$\mathbf{I}(\mathcal{L}) = \boldsymbol{\phi}_{21}(\mathcal{L})\mathbf{E}_0 + [\boldsymbol{\phi}_{22}(\mathcal{L}) - \boldsymbol{\phi}_{21}(\mathcal{L})\mathbf{R}_0]\mathbf{I}(0). \tag{11b}$$

Note that in this formulation, only n simultaneous equations in n unknowns need be solved (11a) requiring $n^3/3$ operations with Gaussian elimination as opposed to $8n^3/3$ operations to solve (7). Of course since (7) is in a partitioned form, one can derive n simultaneous equations to solve in terms of either $\boldsymbol{\alpha}^+$ or $\boldsymbol{\alpha}^-$ but formation of these equations will require the inversion of an $n \times n$ matrix. The voltages at the ends of the line $\mathbf{V}(0)$ and $\mathbf{V}(\mathcal{L})$ are obtainable from the boundary conditions of (2) once the currents at the ends of the line $\mathbf{I}(0)$ and $\mathbf{I}(\mathcal{L})$ are obtained from (11). Furthermore, the voltages and currents at any point on the line $\mathbf{V}(x)$ and $\mathbf{I}(x)$ can be obtained from (8) by replacing $\mathcal{L}$ with x once $\mathbf{I}(0)$ is obtained from (11a) and $\mathbf{V}(0)$ is obtained from (2a).

The matrix chain parameter formulation has an additional advantage. It allows us to consider transmission lines illuminated by an incident EM field as a straightforward extension of the above results. We represent the effects of the spectral components of the incident field at a radian frequency ω as distributed sources along the line so that $n \times 1$ complex-valued source vectors $\mathbf{V}_s(x)$ and $\mathbf{I}_s(x)$ are incorporated into (1) as

$$\begin{bmatrix} \dot{\mathbf{V}}(x) \\ \dot{\mathbf{I}}(x) \end{bmatrix} = -j\omega \begin{bmatrix} {}_n0_n & \mathbf{L} \\ \mathbf{C} & {}_n0_n \end{bmatrix} \begin{bmatrix} \mathbf{V}(x) \\ \mathbf{I}(x) \end{bmatrix} + \begin{bmatrix} \mathbf{V}_s(x) \\ \mathbf{I}_s(x) \end{bmatrix}. \tag{12}$$

The determination of these equivalent sources is generally a difficult matter [1]. One might wish to use the approximation that the source vectors for each line would be determined by sequentially considering the field to illuminate only the reference conductor and the ith conductor for $i = 1,\cdots,n$ and the solution for $[\mathbf{V}_s(x)]_i = V_{s_i}(x)$ and $[\mathbf{I}_s(x)]_i = I_{s_i}(x)$ can be obtained as in [2]. Other approximations are considered in [1].

The solution to (12) can be obtained quite easily by analogy to the solution of state variable equations encountered in automatic control and electrical circuit formulations [4] as

$$\begin{bmatrix} \mathbf{V}(\mathcal{L}) \\ \mathbf{I}(\mathcal{L}) \end{bmatrix} = \boldsymbol{\phi}(\mathcal{L}) \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} + \int_0^\mathcal{L} \boldsymbol{\phi}(\mathcal{L} - \hat{x}) \begin{bmatrix} \mathbf{V}_s(\hat{x}) \\ \mathbf{I}_s(\hat{x}) \end{bmatrix} d\hat{x}$$
$$= \boldsymbol{\phi}(\mathcal{L}) \begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{V}}_s(\mathcal{L}) \\ \hat{\mathbf{I}}_s(\mathcal{L}) \end{bmatrix} \tag{13}$$

and $\boldsymbol{\phi}(\mathcal{L})$ is the solution to the homogeneous set of equations in (1) and given in (8)–(10) [4].

By defining from (13) and (8)

$$\hat{\mathbf{V}}_s(\mathcal{L}) = \int_0^\mathcal{L} \{\boldsymbol{\phi}_{11}(\mathcal{L} - \hat{x})\mathbf{V}_s(\hat{x}) + \boldsymbol{\phi}_{12}(\mathcal{L} - \hat{x})\mathbf{I}_s(\hat{x})\}\, d\hat{x} \tag{14a}$$

$$\hat{\mathbf{I}}_s(\mathcal{L}) = \int_0^\mathcal{L} \{\boldsymbol{\phi}_{21}(\mathcal{L} - \hat{x})\mathbf{V}_s(\hat{x}) + \boldsymbol{\phi}_{22}(\mathcal{L} - \hat{x})\mathbf{I}_s(\hat{x})\}\, d\hat{x} \tag{14b}$$

we can modify (11a) and (11b) to include these source functions quite obviously by simply adding $\hat{\mathbf{I}}_s(\mathcal{L})$ to the right-hand side of (11b) and adding $-\mathbf{V}_s(\mathcal{L}) + \mathbf{R}_\mathcal{L}\hat{\mathbf{I}}_s(\mathcal{L})$ to the right-hand side of (11a). This is quite obvious since (13) shows that $\mathbf{I}(\mathcal{L})$ is increased by $\hat{\mathbf{I}}_s(\mathcal{L})$ and $\hat{\mathbf{V}}(\mathcal{L})$ is increased by $\hat{\mathbf{V}}_s(\mathcal{L})$. Since from (2b) $\mathbf{E}_\mathcal{L} = \mathbf{V}(\mathcal{L}) - \mathbf{R}_\mathcal{L}\mathbf{I}(\mathcal{L})$, then $\mathbf{E}_\mathcal{L}$ on the right-hand side of (11a) is to be decreased by $\hat{\mathbf{V}}_s(\mathcal{L}) - \mathbf{R}_\mathcal{L}\hat{\mathbf{I}}_s(\mathcal{L})$. Thus (11a) and (11b) are modified to consider incident fields by adding additional appropriate forcing functions to their right-hand sides.

II. CABLES IN INHOMOGENEOUS MEDIA

Cable bundles are generally constructed of stranded wires coated with a dielectric material to provide insulation. This inhomogeneity in the medium surrounding the wires precludes the existence of the TEM mode due to the different phase velocities in the different media (free space and insulation dielectric). However, we will consider the TEM mode formulation to be applicable as an approximation similar to the approximation of including losses. This particular problem also has application in predicting responses of flat flexible cable and woven cables which are finding increased useage [7].

The per unit length matrices $\mathbf{L}$ and $\mathbf{C}$ will be symmetric and one can show that $\mathbf{C}$ will be hyperdominant and therefore positive definite [3], [5]. In this case, it is always possible to diagonalize $\mathbf{CL}$ via an $n \times n$ real similarity transformation matrix $\mathbf{T}$ as in (5) and moreover, this transformation will be of a numerically stable type which is a quite necessary attribute for machine computation. Since $\mathbf{C}$ is real symmetric, then there exists an $n \times n$ real symmetric orthogonal transformation matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{C}\mathbf{U} = \boldsymbol{\Lambda}$ where $\boldsymbol{\Lambda}$ is an $n \times n$ real diagonal matrix and $\mathbf{U}^{-1} = \mathbf{U}^T$ (we denote the transpose of a matrix $\mathbf{M}$ by $\mathbf{M}^T$). This is of course a well-known result and is found in almost any text on matrix analysis. Furthermore, since $\mathbf{C}$ is positive definite, the eigenvalues of $\mathbf{C}$, which are the elements of the diagonal matrix $\boldsymbol{\Lambda}$, are all positive (and real). Thus we can quite easily (and meaningfully) form the square root of the matrix $\boldsymbol{\Lambda}$, $\boldsymbol{\Lambda}^{1/2}$, and write

$$\mathbf{A}^{-1/2}\mathbf{U}^{-1}\mathbf{C}\mathbf{U}\mathbf{A}^{-1/2}\mathbf{A}^{1/2}\mathbf{U}^{-1}\mathbf{L}\mathbf{U}\mathbf{A}^{1/2} = \mathbf{A}^{1/2}\mathbf{U}^{-1}\mathbf{L}\mathbf{U}\mathbf{A}^{1/2}$$

but $\mathbf{U}^{-1} = \mathbf{U}^T$ so that $\mathbf{A}^{1/2}\mathbf{U}^T\mathbf{L}\mathbf{U}\mathbf{A}^{1/2}$ is real symmetric and can be diagonalized by an $n \times n$ real symmetric orthogonal transformation $\mathbf{S}$ such that

$$\mathbf{S}^T\mathbf{A}^{1/2}\mathbf{U}^T\mathbf{L}\mathbf{U}\mathbf{A}^{1/2}\mathbf{S} = \gamma^2.$$

Thus we can identify $\mathbf{T}$ in (5) as

$$\mathbf{T} = \mathbf{U}\mathbf{A}^{1/2}\mathbf{S} \tag{15}$$

and it is a simple matter to verify that

$$\mathbf{T}^{-1} = \mathbf{T}^T\mathbf{C}^{-1}. \tag{16}$$

The subroutine NROOT in the IBM Scientific Subroutine Package (SSP) will perform this reduction.

We can cast (11a) and (11b) for this case into an attractive computational form by defining auxiliary variables or mode currents as $\mathbf{I}(x) = \mathbf{T}\mathbf{I}_m(x)$. Then (11) can be written as

$$[\mathbf{R}_\mathcal{L}^*\mathbf{E}^+(\mathcal{L}) + \mathbf{R}_\mathcal{L}^*\mathbf{E}^-(\mathcal{L})\gamma^{-1}\mathbf{R}_0^* + \gamma\mathbf{E}^-(\mathcal{L}) + \mathbf{E}^+(\mathcal{L})\mathbf{R}_0^*]\mathbf{I}_m(0)$$
$$= +\mathbf{V}_s^*(\mathcal{L}) - \mathbf{R}_\mathcal{L}^*\mathbf{I}_s^*(\mathcal{L}) + [\mathbf{E}^+(\mathcal{L}) + \mathbf{R}_\mathcal{L}^*\mathbf{E}^-(\mathcal{L})\gamma^{-1}]\mathbf{E}_0^* - \mathbf{E}_\mathcal{L}^* \tag{17a}$$

$$\mathbf{I}_m(\mathcal{L}) = [\mathbf{E}^+(\mathcal{L}) + \mathbf{E}^-(\mathcal{L})\gamma^{-1}\mathbf{R}_0^*]\mathbf{I}_m(0) - \mathbf{E}^-(\mathcal{L})\gamma^{-1}\mathbf{E}_0^* + \mathbf{I}_s^*(\mathcal{L}). \tag{17b}$$

To obtain (17) we simply substitute the change of variables $\mathbf{I}(x) = \mathbf{T}\mathbf{I}_m(x)$ along with (8)–(10) into (11) and premultiply (11a) by $\mathbf{T}^T$ and (11b) by $\mathbf{T}^{-1} = \mathbf{T}^T\mathbf{C}^{-1}$ [the identity in (16)]. The algebra is straightforward and omitted here. We also find it convenient from a notational standpoint to define $\mathbf{V}_s^*(\mathcal{L}) = \mathbf{T}^T\hat{\mathbf{V}}_s(\mathcal{L})$ and $\mathbf{I}_s^*(\mathcal{L}) = \mathbf{T}^T\mathbf{C}^{-1}\hat{\mathbf{I}}_s(\mathcal{L})$. Similarly, we have defined $\mathbf{R}_0^* = \mathbf{T}^T\mathbf{R}_0\mathbf{T}$, $\mathbf{R}_\mathcal{L}^* = \mathbf{T}^T\mathbf{R}_\mathcal{L}\mathbf{T}$, $\mathbf{E}_0^* = \mathbf{T}^T\mathbf{E}_0$, and $\mathbf{E}_\mathcal{L}^* = \mathbf{T}^T\mathbf{E}_\mathcal{L}$ for notational convenience. It is quite natural to do this since defining mode currents as $\mathbf{I}(x) = \mathbf{T}\mathbf{I}_m(x)$ and mode voltages as $\mathbf{V}_m(x) = \mathbf{T}^T\mathbf{V}(x)$ it is clear from (6) [and using the identity in (16)] that the modes, $\mathbf{I}_m(x)$ and $\mathbf{V}_m(x)$, consist of $2n$ uncoupled waves and the boundary conditions of (2) in terms of mode quantities become

$$\mathbf{V}_m(0) = \mathbf{E}_0^* - \mathbf{R}_0^*\mathbf{I}_m(0) \tag{18a}$$

$$\mathbf{V}_m(\mathcal{L}) = \mathbf{E}_\mathcal{L}^* + \mathbf{R}_\mathcal{L}^*\mathbf{I}_m(\mathcal{L}). \tag{18b}$$

Thus $\mathbf{E}_0^*$ and $\mathbf{E}_\mathcal{L}^*$ are the equivalent sources for the modes and $\mathbf{R}_0^*$ and $\mathbf{R}_\mathcal{L}^*$ are the resistive terminations for the modes.

For numerical computation, (17a) requires $n^3/3$ operations to solve with n^2 operations required to form $\mathbf{I}(0) = \mathbf{T}\mathbf{I}_m(0)$. In addition, forming (17) is straightforward since $\mathbf{E}^+(\mathcal{L})$, $\mathbf{E}^-(\mathcal{L})$, and γ are diagonal matrices. Also γ^{-1} is easily determined since γ is diagonal and the determination of $\mathbf{V}_s^*(\mathcal{L})$ and $\mathbf{I}_s^*(\mathcal{L})$ on the right-hand side of (17a) is also quite easily determined. For example, from (14), (9), and (10) and utilizing the relation $\mathbf{T}^{-1} = \mathbf{T}^T\mathbf{C}^{-1}$ in (16)

$$\mathbf{V}_s^*(\mathcal{L}) = \mathbf{T}^T\int_0^\mathcal{L}\{\mathbf{C}^{-1}\mathbf{T}\mathbf{E}^+(\mathcal{L}-\hat{x})\mathbf{T}^{-1}\mathbf{C}\mathbf{V}_s(\hat{x})$$
$$- \mathbf{C}^{-1}\mathbf{T}\gamma\mathbf{E}^-(\mathcal{L}-\hat{x})\mathbf{T}^{-1}\mathbf{I}_s(\hat{x})\}\,d\hat{x}$$
$$= \int_0^\mathcal{L}\{\mathbf{E}^+(\mathcal{L}-\hat{x})\mathbf{T}^T\mathbf{V}_s(\hat{x}) - \gamma\mathbf{E}^-(\mathcal{L}-\hat{x})\mathbf{T}^T\mathbf{C}^{-1}\mathbf{I}_s(\hat{x})\}\,d\hat{x} \tag{19a}$$

$$\mathbf{I}_s^*(\mathcal{L}) = \mathbf{T}^T\mathbf{C}^{-1}\int_0^\mathcal{L}\{-\mathbf{T}\mathbf{E}^-(\mathcal{L}-\hat{x})\gamma^{-1}\mathbf{T}^{-1}\mathbf{C}\mathbf{V}_s(\hat{x})$$
$$+ \mathbf{T}\mathbf{E}^+(\mathcal{L}-\hat{x})\mathbf{T}^{-1}\mathbf{I}_s(\hat{x})\}\,d\hat{x}$$
$$= \int_0^\mathcal{L}\{-\mathbf{E}^-(\mathcal{L}-\hat{x})\gamma^{-1}\mathbf{T}^T\mathbf{V}_s(\hat{x})$$
$$+ \mathbf{E}^+(\mathcal{L}-\hat{x})\mathbf{T}^T\mathbf{C}^{-1}\mathbf{I}_s(\hat{x})\}\,d\hat{x}. \tag{19b}$$

Normally, one determines $\mathbf{C}$ by writing the potentials on the conductors in terms of the charges on the conductors as $\mathbf{V} = \mathbf{C}^{-1}\mathbf{Q}$ and thus it is unnecessary to invert $\mathbf{C}$.

Therefore the equations which require the majority of the computational expense, (11a) which for this case is (17a), can be solved with perhaps minimal effort as opposed to other approaches where one does not condition the equations, i.e., cast them into an equivalent but more desirable computational form.

III. CABLES IN HOMOGENEOUS MEDIA

The current trend on many modern avionics systems is to route cable bundles in conduit to protect them from incident field effects. One may wish to approximate this situation as n conductors imbedded in a homogeneous dielectric whose relative permittivity is that of the wire insulation or some measured effective relative permittivity [1] and we may wish to determine the effects of currents induced on the conduit [the $(n+1)$st conductor] by the incident field when portions of the conduit pass apertures.

Here we may utilize the important fact that for a homogeneous medium

$$\mathbf{L}\mathbf{C} = \mathbf{C}\mathbf{L} = \frac{1}{v^2}\mathbf{I}_n \tag{20}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix and v is the phase velocity in the surrounding medium $v = 1/(\mu\epsilon)^{1/2}$ [8]. In this case, the matrix chain parameters become [$\mathbf{T} = \mathbf{I}_n$ and $\gamma^2 = 1/v^2\mathbf{I}_n$ in (5)]

$$\phi_{11}(\mathcal{L}) = \tfrac{1}{2}[e^+(\mathcal{L}) + e^-(\mathcal{L})]\mathbf{I}_n$$
$$\phi_{12}(\mathcal{L}) = -\tfrac{1}{2}[e^+(\mathcal{L}) - e^-(\mathcal{L})]\mathbf{Z}_C$$
$$\phi_{21}(\mathcal{L}) = -\tfrac{1}{2}[e^+(\mathcal{L}) - e^-(\mathcal{L})]\mathbf{Z}_C^{-1}$$
$$\phi_{22}(\mathcal{L}) = \tfrac{1}{2}[e^+(\mathcal{L}) + e^-(\mathcal{L})]\mathbf{I}_n \tag{21}$$

where the complex scalars $e^+(\mathcal{L})$ and $e^-(\mathcal{L})$ are given by $e^+(\mathcal{L}) = e^{j\omega\mathcal{L}/v}$ and $e^-(\mathcal{L}) = e^{-j\omega\mathcal{L}/v}$, and the characteristic impedance matrix is given by $\mathbf{Z}_C = v\mathbf{L}$ [3], [5].

From (14), we obtain

$$\hat{\mathbf{V}}_s(\mathcal{L}) = \int_0^\mathcal{L}\{\tfrac{1}{2}[e^+(\mathcal{L}-\hat{x}) + e^-(\mathcal{L}-\hat{x})]\mathbf{V}_s(\hat{x})$$
$$- \tfrac{1}{2}[e^+(\mathcal{L}-\hat{x}) - e^-(\mathcal{L}-\hat{x})]\mathbf{Z}_C\mathbf{I}_s(\hat{x})\}\,d\hat{x} \tag{22a}$$

$$\hat{\mathbf{I}}_s(\mathcal{L}) = \int_0^\mathcal{L}\{-\tfrac{1}{2}[e^+(\mathcal{L}-\hat{x}) - e^-(\mathcal{L}-\hat{x})]\mathbf{Z}_C^{-1}\mathbf{V}_s(\hat{x})$$
$$+ \tfrac{1}{2}[e^+(\mathcal{L}-\hat{x}) + e^-(\mathcal{L}-\hat{x})]\mathbf{I}_s(\hat{x})\}\,d\hat{x}. \tag{22b}$$

Incorporating the boundary conditions, it is a straightforward matter to derive from (11) with ϕ_{ij} in (21) and including $\mathbf{V}_s(\mathcal{L})$ and $\hat{\mathbf{I}}_s(\mathcal{L})$ as described previously

$$[\alpha(\mathbf{R}_\mathcal{L} + \mathbf{R}_0) + \beta(\mathbf{R}_\mathcal{L}\mathbf{Z}_C^{-1}\mathbf{R}_0 + \mathbf{Z}_C)]\mathbf{I}(0)$$
$$= \gamma(-\hat{\mathbf{V}}_s(\mathcal{L}) + \mathbf{R}_\mathcal{L}\hat{\mathbf{I}}_s(\mathcal{L})) + \gamma\mathbf{E}_\mathcal{L} + [\alpha\mathbf{I}_n + \beta\mathbf{R}_\mathcal{L}\mathbf{Z}_C^{-1}]\mathbf{E}_0 \tag{23a}$$

$$\mathbf{I}(\mathcal{L}) = \frac{\beta}{\gamma}\mathbf{Z}_C^{-1}\mathbf{E}_0 - \left[\frac{\alpha}{\gamma}\mathbf{I}_n + \frac{\beta}{\gamma}\mathbf{Z}_C^{-1}\mathbf{R}_0\right]\mathbf{I}(0) + \hat{\mathbf{I}}_s(\mathcal{L}) \tag{23b}$$

where the complex scalars α, β, and γ are defined as $\alpha = (e^2 + 1)$, $\beta = (e^2 - 1)$, $\gamma = -2e^+$, and $e^2 = (e^+)^2$.

Again the major computational effort is consumed in solving the n equations in n unknowns in (23a). Since $\mathbf{R}_\mathcal{L}$ and $\mathbf{R}_0$ are symmetric positive definite, then one can show that $(\mathbf{R}_0 + \mathbf{R}_\mathcal{L})$ will be symmetric positive definite and therefore nonsingular. Thus we may form by premultiplying (23a) by $(\mathbf{R}_\mathcal{L} + \mathbf{R}_0)^{-1}$

$$[\alpha\mathbf{I}_n + \beta(\mathbf{R}_\mathcal{L} + \mathbf{R}_0)^{-1}(\mathbf{R}_\mathcal{L}\mathbf{Z}_C^{-1}\mathbf{R}_0 + \mathbf{Z}_C)]\mathbf{I}(0)$$
$$= \gamma(\mathbf{R}_\mathcal{L} + \mathbf{R}_0)^{-1}[-\hat{\mathbf{V}}_s(\mathcal{L}) + \mathbf{R}_\mathcal{L}\hat{\mathbf{I}}_s(\mathcal{L})] + \gamma(\mathbf{R}_\mathcal{L} + \mathbf{R}_0)^{-1}\mathbf{E}_\mathcal{L}$$
$$+ (\mathbf{R}_\mathcal{L} + \mathbf{R}_0)^{-1}[\alpha\mathbf{I}_n + \beta\mathbf{R}_\mathcal{L}\mathbf{Z}_C^{-1}]\mathbf{E}_0. \tag{24}$$

If we can find an $n \times n$ nonsingular matrix $\mathbf{M}$ such that a change of variables $\mathbf{I}(0) = \mathbf{M}\mathbf{I}^*(0)$ in (24) yields

$$\mathbf{M}^{-1}[(\mathbf{R}_\mathcal{L} + \mathbf{R}_0)^{-1}(\mathbf{R}_\mathcal{L}\mathbf{Z}_C^{-1}\mathbf{R}_0 + \mathbf{Z}_C)]\mathbf{M} = \boldsymbol{\theta} \tag{25}$$

and $\boldsymbol{\theta}$ is an $n \times n$ diagonal matrix, then the equations in (24) in terms of $\mathbf{I}^*(0)$ will be uncoupled requiring a trivial amount of computation to solve for $\mathbf{I}^*(0)$ and only n^2 operations to recover $\mathbf{I}(0)$ through $\mathbf{I}(0) = \mathbf{M}\mathbf{I}^*(0)$. Thus if it is possible to uncouple the equations in this manner, then we do not need to solve large numbers of simultaneous equations at each frequency. Unfortunately, it is not always possible to uncouple the equations as the following example shows:

$$\mathbf{R}_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{R}_{\mathcal{L}} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad Z_C = \begin{bmatrix} 5^{1/2} & 2 \\ 2 & 5^{1/2} \end{bmatrix}. \tag{26}$$

Even if under certain cases it is possible, theoretically, to diagonalize the coefficient matrix in (24), such a transformation can easily lead to numerical instabilities unless it is of a numerically stable type such as an orthogonal transformation [6].

Although it is not always possible to obtain $\boldsymbol{\theta}$ in (25) in diagonal form, it is always possible to find a very stable transformation $\mathbf{M}$ such that $\boldsymbol{\theta}$ is in lower (or upper) Hessenberg form [6]. The lower Hessenberg form of $\boldsymbol{\theta}$ is such that a large number of the entries in $\boldsymbol{\theta}$ are zero, i.e., $[\boldsymbol{\theta}]_{ij} = 0$, $i = 1,\cdots,n-2$, and $j = (i+2),\cdots,n$. Thus the Hessenberg form is in "almost" lower triangular form.

Gaussian elimination utilizes row operations to reduce the coefficient matrix to lower triangular form and then back substitution is utilized to find the solutions. The majority of the operations are consumed in the reduction to lower triangular form.

With the transformation to Hessenberg form, (24) becomes

$$\begin{aligned}[\alpha \mathbf{I}_n + \beta\boldsymbol{\theta}]\mathbf{I}^*(0) &= \gamma\mathbf{M}^{-1}(\mathbf{R}_{\mathcal{L}} + \mathbf{R}_0)^{-1}[-\hat{\mathbf{V}}_s(\mathcal{L}) + \mathbf{R}_{\mathcal{L}}\hat{\mathbf{I}}_s(\mathcal{L})] \\ &\quad + \gamma\mathbf{M}^{-1}(\mathbf{R}_{\mathcal{L}} + \mathbf{R}_0)^{-1}\mathbf{E}_{\mathcal{L}} + \mathbf{M}^{-1}(\mathbf{R}_{\mathcal{L}} + \mathbf{R}_0)^{-1} \\ &\quad \cdot[\alpha\mathbf{I}_n + \beta\mathbf{R}_{\mathcal{L}}Z_C^{-1}]\mathbf{E}_0. \end{aligned} \tag{27}$$

Then one can employ row operations to reduce (27) to lower triangular form with back substitution being utilized to solve for the elements of $\mathbf{I}^*(0)$. $\mathbf{I}(0)$ can then be obtained from $\mathbf{I}(0) = \mathbf{MI}^*(0)$.

Solving (24) with Gaussian elimination and back substitution requires on the order of $n^3/3$ per-frequency operations for large n. Solution via the reduction to Hessenberg form [solution of (27)] requires only $[n^2/2 + n/3]$ operations for triangularization, $[n^2/2 + n/2]$ operations for back substitution, and n^2 operations to form $\mathbf{I}(0) = \mathbf{MI}^*(0)$ so that the total number of per-frequency operations has been reduced from on the order of $n^3/3$ with Gaussian elimination to $2n^2 + n$ for the Hessenberg reduction; a substantial savings for large n. Furthermore, the reduction to Hessenberg form is frequency independent and only needs to be performed once at the beginning of the frequency iteration.

If each line is connected to the reference conductor only through a single resistance (a very common situation), then $\mathbf{R}_0$ and $\mathbf{R}_{\mathcal{L}}$ will be diagonal and $(\mathbf{R}_0 + \mathbf{R}_{\mathcal{L}})^{-1}$ is trivial to obtain. $\mathbf{M}^{-1}$ is quite simple to obtain as a sequence of row operations [6] so that formation of (27) is not really so difficult.

Thus we are able to reduce the number of per-frequency operations in the homogeneous medium case from on the order of n^3 to on the order of n^2—a substantial savings for large n.

IV CONCLUSION

Numerically efficient methods of computing the frequency response of multiconductor transmission lines in homogeneous and inhomogeneous media illuminated by an EM field are presented. The formulations allow an efficient determination of the frequency response for cables consisting of a large number of coupled conductors with various port load conditions. The transformations used are numerically stable with respect to roundoff error and are frequency independent so that they need be determined only once at the beginning of the frequency iteration.

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On the Calibration Process of Automatic Network Analyzer Systems

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Abstract—**Formulas are presented for the direct calculation of the scattering parameters of a linear two-port, when it is measured by an imperfect network analyzer. Depending on the hardware configuration of the test set, the measurement system is represented by one of two flowgraph models. In both models presented, leakage paths are included. The error parameters, i.e., the scattering parameters of the measuring system, are six respective ten complex numbers for each frequency of interest. A calibration procedure, where measurements are made on standards, will determine the error parameters. One of many possible calibration procedures is briefly described. By using explicit formulas instead of iterative methods, the computing time for the correction of the scattering parameters of the unknown two-port is significantly reduced. The addition of leakage paths will only have a minor effect on computational complexity while measurement accuracy will increase.**

An important property of automatic network analyzers is that system errors can be brought to a minimum by a calibration process [1]. Two different measuring systems, represented by flowgraph models, will be considered in this short paper. Fig. 1 shows a schematic of the hardware configuration, with the digital computer excluded.

Which model to apply depends on whether the coaxial switch S_a is used or not. If the switch S_a is not included, the device under test has to be manually turned to be measured from both directions. In this case, the flowgraph model presented by Hand [2] is applicable. This model is shown in Fig. 2.

s_{11}, s_{12}, s_{21}, and s_{22} are the scattering parameters of the device under test. e_{00}–e_{32} are parameters representing errors in the system. By making measurements on standards, the error parameters can be determined. Three reflexion measurements with $s_{21} = 0$ are enough to determine e_{00}, e_{01}, and e_{11}. This can be done with a perfect termination, a direct and an offset short. A sliding load can simulate the perfect termination. A transmission measurement with $s_{21} = 0$ will give e_{30}. e_{22} and e_{32} can then be determined if $s_{21} = s_{12} = 1$ and $s_{11} = s_{22} = 0$, i.e., a through connection. A thorough description of the calibration process is given in [2]. Another similar calibration method is described in [3].

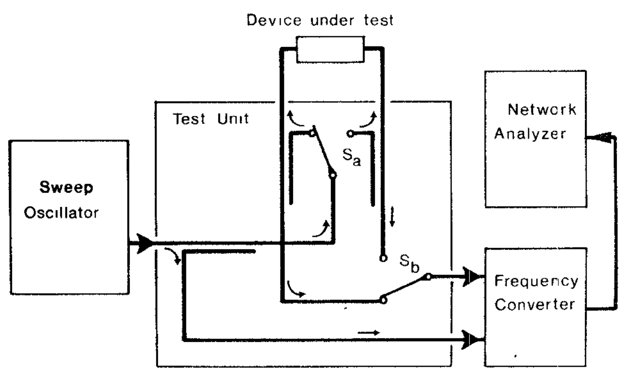

Fig. 1. Hardware configuration.

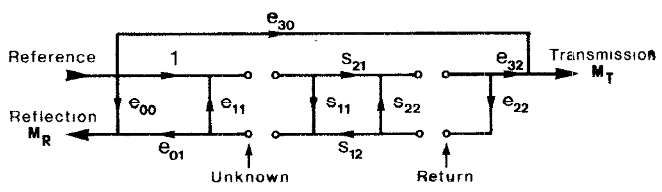

Fig. 2. Signal flowgraph of system model (switch S_a not included in test unit).

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